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OPTIMIZATION OF CONTROL PARAMETERS OF FLOWS WITH TRANSITION THROUGH THE SPEED OF SOUND*

F.A. SLOBODKINA and E.K. IANOVSKAIA

The problem of optimization of parameters controlling the transonic flow in channels in the presence of external effects is studied with the application of the method of Lagrange's multipliers. The gasdynamic flow is defined by quasi-one dimensional equations. Among various modes steady flows are selected with continuous transition through the speed of sound from subsonic velocity to supersonic, and vice versa. The selection of steady solutions with passing through the speed of sound is carried out on the basis of criteria derived in /1,2/. Solution of the problem of optimization of transonic flows in MHD-generator and in MHD-accelerator is given at small Reynolds numbers and in a constant current plasmotron with length-wise blown arc.

1. We consider the flow of gas in a quasi-one-dimensional approximation in a finite length $l\,(0 \leqslant x \leqslant l)$ channel of a varying cross section $y(x)$ in the presence of external effects. We write the equations of continuity, and motion and energy of unsteady quasi-one-dimensional flow of perfect gas in the form

$$\begin{aligned}
&L_1 \equiv \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} + \rho u\frac{y'}{y} + f_1(\rho, u, p, v_k) = 0 \\
&L_2 \equiv \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} + f_2(\rho, u, p, v_k) = 0 \\
&L_3 \equiv \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + \gamma p\frac{\partial u}{\partial x} + \gamma p u\frac{y'}{y} + f_3(\rho, u, p, v_k) = 0 \\
&y' = \frac{dy}{dx}, \quad h = \frac{\gamma}{\gamma - 1}\frac{p}{\rho}, \quad k = 1, 2, \ldots, m
\end{aligned} \tag{1.1}$$

where ρ, u, p, h are the density, velocity, pressure, and enthalpy of the medium, respectively, γ is the ratio of specific heats; functions f_1, f_2, f_3 define the external effects (mass addition and removal, various energy forms, etc.), $y(x)$ and $v_k(x)$ are the system control parameters, the x axis coincides with the channel axis and the gas moves in the positive direction of the x axis.

To formulate the boundary value problem on finite segment of the x axis for the system of Eqs.(1.1), we calculate the characteristic velocities, which are determined by the equations

$$[(u - c)^2 - a^2](u - c) = 0, \quad a^2 = \gamma p/\rho \tag{1.2}$$

These velocities are $c^{(1)} = u - a$, $c^{(2)} = u + a$, $c^{(3)} = u$.

Since from the physical point of view only that part of space where the flow parameters ρ, u, p are positive and finite, are of interest, in conformity with (1.2) we can state that the system of Eqs.(1.1) are of the hyperbolic type when $0 < \rho, u, p < \infty$.

The characteristic velocities $c^{(2)}, c^{(3)}$ are always positive in the indicated above variation region of variables x, u, ρ, p, while $c^{(1)} = u - a$ may be either positive, or negative, or change its sign at some points $x \in [0, l]$. Subsequently we shall assume the existence at least of one inner point $x \in (0, l)$ where $c^{(1)}$ changes it sign, but not excluding the possibility of other such points coincident with the ends of segment $[0, l]$.

The behavior of characteristic velocity $c^{(1)}$ for $0 \leqslant x \leqslant l$ defines the flow mode. The flow over the whole channel length is supersonic when $c^{(1)} > 0$, subsonic when $c^{(1)} < 0$, and transonic when $c^{(1)}$ changes its sign inside the channel. In the latter case $c^{(1)}$ can change its sign either in a continuous manner, or discontinuously in a shock wave.

Among various types of transonic flows we shall consider steady flows with continuous transition through the speed of sound. Steady flows are defined by a system of ordinary differential equations which are obtained from (1.1) with time derivatives equal zero. The point of transition through the speed of sound corresponds to a singular point of steady equations.

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As shown by the investigation of an arbitrary hyperbolic system of equations, whose unknown functions depend on two arguments, viz. coordinate and time /1,2/, we can separate three types of stable steady solutions of system (1.1) with a continuous transition through the speed of sound. We shall consider, as in /1,2/, the transonic solution as stable for which the presence of point of transition through the speed of sound does not lead to the development of instability.

Solutions with transition through the speed of sound from supersonic to subsonic velocity (from $c^{(1)} > 0$ to $c^{(1)} < 0$) are considered stable, if the singular point of steady equations (1.1) is a node with negative proper directions, while solutions with transition from subsonic to supersonic velocity (from $c^{(1)} < 0$ to $c^{(1)} > 0$) is considered stable when the singular point is a saddle or a deformed saddle. In the latter case the derivatives of flow parameters with respect to x become infinite /1,2/.

Under certain conditions obtained in /1,2/, solutions with transition through the speed of sound from $c^{(1)} > 0$ to $c^{(1)} < 0$ at the singular point of the saddle type can also be stable, but they are not considered here.

As shown in /1,2/, the system of equations of the hyperbolic type close to the singular point reduces to a single first order differential equation. The type of singular point of that equation and, consequently of the whole system, with time derivatives equal zero, is determined by two eigenvalues λ_1, λ_2. When the problem controlling functions are continuous, singular points of the node and saddle type may be generated, and when they are discontinuous, singularities of the deformed saddle may appear.

For correctly stating the mixed problem for a system of Eqs.(1.1) and, consequently also, for the correct statement of the boundary value problem for the respective stationary system, it is necessary to know the type of the considered transonic flow.

In the presence of singularity of the node type $(\lambda_1 < 0, \lambda_2 < 0)$ it is necessary to specify inside the x segment three conditions at $x = 0$, since at that end $c^{(1)} > 0, c^{(2)} > 0, c^{(3)} > 0$, and one condition at $x = l$, since at that boundary $c^{(1)} < 0$.

In the presence of a singularity of the saddle type $(\lambda_1 > 0, \lambda_2 < 0)$ or of the deformed saddle type for solution with transition from $c^{(1)} < 0$ to $c^{(1)} > 0$ it is necessary to specify only two conditions at $x = 0$, since in that case with $x = 0$ only $c^{(2)}$ and $c^{(3)}$ are positive ($c^{(1)}$ is negative), and at $x = l$ all three characteristic velocities are positive and, consequently, boundary conditions at $x = l$ are not specified.

Thus in the first case the steady system consisting of three first order differential equations is integrated with four boundary conditions, and in the second and third cases, with two. The possibility of selecting a solution that correspond to equations and boundary conditions is determined in the first case by that one-parameter set of integral curves passes through the singular point of the node type, and in the second and third cases the selection of solution is dependent on the supplementary condition of the integral curve passing through the singular point with the positive derivative $dc^{(1)}/dx$.

The form of boundary conditions is determined by the statement of a particular problem, and must ensure the respective type of the considered transonic flow.

2. Let us formulate the problem of selecting the control functions $y(x), v_k(x)$ so that some integral characteristic

$$N = \int_0^{x_b} \Phi(x, y, y', v_k, \rho, u, p, \ldots)\, dx \tag{2.1}$$

reaches the maximum value, where Φ is a known function of its arguments.

When solving the problem of optimization, it is necessary to take into account that the control function must satisfy certain limitations.

We assume that the admissible control $v_k(x)$ and the derivative of the channel form $y'(x)$ are piecewise continuous functions that can suffer a first order discontinuity at a finite number of points of the segment of x, and functions $y(x)$ and $\rho(x), u(x), p(x)$ are continuous.

Let the controls $v_k(x)$ and $y'(x)$ satisfy the inequantities

$$-k_1 \leqslant y' \leqslant k_2, \quad V_{1k} \leqslant v_k \leqslant V_{2k} \tag{2.2}$$

the first of which is associated with the boundaries of quasi-one-dimensional approximation and the second with the technical possibilities of the controls themselves (the constants k_1, k_2 are positive).

The solution of the optimization problem is carried out with the use of variable Lagrange multipliers. For this we compose the subsidiary functional

$$I = N + \int_0^{x_b} (\mu_1 L_1 + \mu_2 L_2 + \mu_3 L_3)\, dx \tag{2.3}$$

where $\mu_i(x)$ the variable Lagrange multipliers and the expressions L_1, L_2, L_3 are steady.

Calculating the first variation δI taking into account the appearance of point of control discontinuity and the change of sign of $c^{(1)}$ we obtain equations for the Lagrange multipliers and the boundary conditions for their integration and, also, the necessary conditions of extremum for all control functions /4/.

We draw the attention on that in optimizing a flow with singularity of the node type $(\lambda_1 < 0, \lambda_2 < 0)$, we obtain for the equations of the Lagrange multipliers two boundary conditions at $x = l$, while in optimizing flows with singularity of the saddle type, or of deformed saddle type for the same equations we obtain three boundary conditions at $x = l$ and one at $x = 0$.

As shown in /4/, the solvability of the formulated boundary value problems follows from the investigation of the properties of the singular point of the system consisting of equations of motion and of equations for the Lagrange multipliers. The singular point of such system proved to be a generalized saddle, at which there always is one and only one proper direction to which is tangent the one-parameter set of integral curves. In the first case the presence of this proper direction is due to the equations of motion, and in the second and third by the equations of the Lagrange multipliers.

3. Recently, the method of establishment is applied effectively to the solution of problems of transonic flows. Here the method of establishment with respect to time is applied to solving the problems of optimizing the controls of flows with transition through the speed of sound.

We write the unsteady equations for the Lagrange multipliers that are conjugate of the unsteady system (1.1)

$$
\begin{aligned}
&\frac{\partial\mu_1}{\partial t} + u\frac{\partial\mu_1}{\partial x} = \Phi_\rho + \mu_1\left(\frac{uy'}{y} + f_{1\rho}\right) + \mu_2\left(f_{2\rho} - \frac{1}{\rho^2}\frac{\partial p}{\partial x}\right) + \mu_3 f_{3\rho} \\
&\frac{\partial\mu_2}{\partial t} + \rho\frac{\partial\mu_1}{\partial x} + u\frac{\partial\mu_2}{\partial x} + \gamma p\frac{\partial\mu_3}{\partial x} = \\
&\qquad \Phi_u + \mu_1\left(\rho\frac{y'}{y} + f_{1u}\right) + \mu_2 f_{2u} + \mu_3\left[f_{3u} - (\gamma-1)\frac{\partial\rho}{\partial x} + \frac{\gamma p y'}{y}\right] \\
&\frac{\partial\mu_3}{\partial t} + \frac{1}{\rho}\frac{\partial\mu_2}{\partial x} + u\frac{\partial\mu_3}{\partial x} = \\
&\qquad \Phi_p + \mu_1 f_{1p} + \mu_2\left(f_{2p} + \frac{1}{\rho^2}\frac{\partial p}{\partial x}\right) + \mu_3\left[\frac{\gamma u y'}{y} + (\gamma-1)\frac{\partial u}{\partial x} + f_{3p}\right]
\end{aligned}
\tag{3.1}
$$

where the subscripts ρ, u, p denote the respective partial derivatives.

The form of system of Eqs.(3.1) implies that its characteristic velocities are the same as of system (1.1).

It can be ascertained that all characteristic velocities of the system composed of equations (1.1) and (3.1) are double. This conclusion additionally clarifies the results of /4/, that generalizes the quantitative analysis carried out in /1/, in the case of system with double characteristic velocity which vanishes.

In obtaining the steady solution of Eqs.(1.1) and (3.1) by the method of establishment, steady boundary conditions and some initial data are specified. The equations of motion are integrated in the direction of increasing time, and those for the Lagrange multipliers in the direction of decreasing time.

Such procedure is dictated by the correct statement of the mixed problem /3/ formulated in Sects.1 and 3 for the hyperbolic systems of Eqs.(1.1) and (3.1).

Let us elucidate this on an example. Let at the inlet cross section of the channel the flow be supersonic and at the outlet subsonic. Then, as already stated, for the equations of motion three conditions must be specified at $x = 0$, since at that end the three characteristic velocities are positive, and one condition at $x = l$, since here the characteristic velocity $c^{(1)}$ is negative. Solution of such problem for system (1.1) with initial conditions $u(0, x)$, $\rho(0, x)$, $p(0, x)$ at $t = 0$ can be obtained when $t > 0$, $0 \leqslant x \leqslant l$.

In that case the boundary value problem for Lagrange multipliers yields two conditions at $x = l$. Because the characteristic velocities $c^{(2)}$ and $c^{(3)}$ of system (3.1) are positive at $x = l$, the mixed problem for the Lagrange multipliers must be solved when $t < 0$, since only for such direction of time the information from the boundary $x = l$ is transmitted with the characteristic velocities $c^{(2)}$ and $c^{(3)}$ to the inside of segment $0 \leqslant x \leqslant l$.

The case when the flow at the channel intake is subsonic and at the outlet supersonic can be analyzed similarly.

Analysis /4/ shows that the character of singular points of system of Eqs.(1.1) and (3.1) for stable transonic flows enables the construction of solution for the Lagrange multipliers.

For solving the problem of optimization we propose the following iteration process: specified are the control functions $y^\circ(x), v_k^\circ(x)$ and the initial distributions $\rho^\circ(0, x)$, $u^\circ(0, x)$, $p^\circ(0, x)$ that satisfy the specified boundary conditions, and Eqs.(1.1) are integrated by the method of establishment up to the emergence of a steady mode. Then the obtained steady distributions of $\rho(x), u(x), p(x)$ are used for calculating the coefficients in Eqs.(3.1) which are then integrated for some initial distribution of $\mu_1^\circ(0, x), \mu_2^\circ(0, x), \mu_3^\circ(0, x)$, also, satisfying their boundary conditions for obtaining steady values of $\mu_i(x)$.

The obtained distribution of $\rho(x), u(x), p(x), \mu_i(x)$ are used for calculating the control parameters with the constraints (2.2), and the process is repeated with the new controls $y^1(x)$, $v_k^1(x)$. The initial distributions for ρ, u, p and later also for μ_i, the respective steady solution obtained in the preceding step is taken. The indicated calculations are repeated until the values of control parameters in two successive iterations do not differ by more than ε (ε is the specified exactness).

Such approach to the solution of the problem of transonic flows proved to be more general than the many other methods using the numerical integration of ordinary differential equations with ranging over the deficient boundary conditions.

Without going into the difficulties of constructing the algorithm using the method of solving boundary value problems for ordinary differential equations with singular points (e.g., /4/), we only note that preliminary qualitative investigations of unsteady equations of motion it is necessary in any case, no matter by which method the equations are integrated, since the type of singular point determines the stability of the flow mode. It is not necessary to conduct in each case a qualitative analysis for the Lagrange multipliers, since the singularity of these equations is defined in /4/ for all types of solutions with transition through zero of characteristic velocities.

4. We adduce examples of optimization of specific transonic flows in channels.

As the first example, let us consider the problem of optimization of parameters of an MHD-generator. A detailed statement and solution of the problem of optimization of MHD-generator with respect to power for subsonic, supersonic and transonic flows with transition through the speed of sound in the shock wave are given in /5/. In /4/ is given the statement and the solution of that problem for flow with transition through the speed of sound in a singular point of the steady node type. Here, we present some supplementary results obtained by the method of establishment.

The flow of a electro-conducting gas in a plane channel of MHD-generator at low magnetic Reynolds numbers is defined by the Eqs.(1.1) under conditions

$$f_1 = 0, \quad f_2 = \Delta\sigma B\left(uB - \frac{\varphi}{y}\right), \quad f_3 = \Delta\sigma\varphi\left(uB - \frac{\varphi}{y}\right), \ \Delta = \frac{B^{\circ 2}\sigma^\circ l^\circ}{\rho^\circ\sqrt{2h^\circ}} \tag{4.1}$$

where σ is the medium electrical conductivity, B is the external magnetic field intensity normal to the plane in which lies the x axis and the generatrices of the conducting channel walls, φ and $-\varphi$ are the potential of the upper and lower wall, respectively, and Δ is the parameter of the magnetohydrodynamic interaction. The equations are in dimensionless variables, as in /5/, and the parameters with small circle superscripts denote characteristic quantities.

Assuming the flow to be supersonic at the inlet whose cross section is fixed, we specify parameters ρ_0, u_0, p_0 at $x = 0$ from the consideration of flow at $x < 0$ which can be assumed known. In the open cycle work of the generator, the pressure p_∞ of the medium into which the outflow takes place is specified. In conformity with /4,5/ that pressure must be such that the discharge occurs at the speed of sound, i.e. $M_b = 1$ (M is the Mach number).

Thus for the system of Eqs.(1.1) and (4.1) the boundary conditions are specified in the form

$$\rho = \rho_0, \quad u = u_0, \quad p = p_0 \ (x = 0), \ M_b = 1 \ (x = x_b) \tag{4.2}$$

Functions $y(x), \varphi(x), B(x)$ are the controls of this function and x_b is the channel length.

Taking the maximum admissible channel length as a characteristic dimension l° and the module of maximum admissible magnetic field intensity as B°, we obtain the inequalities

$$0 \leqslant x \leqslant x_b \leqslant 1, \quad -1 \leqslant B(x) \leqslant 1, \quad \varphi_1 \leqslant \varphi(x) \leqslant \varphi_2, \ y(0) = 1 \tag{4.3}$$

As shown by the qualitative investigation /6/, the flow in widening channel of the MHD-generator is realized with a transition through the speed of sound in a singularity of the node type with negative proper direction for values of parameter $1.5 \leqslant \Delta < \infty$ in the range of change $B(x)$ and $\varphi(x)$ corresponding to (4.3).

As the optimized characteristic we select the power taken off per unit of width of the generator

$$N = \int_0^{x_b} \Delta\sigma\varphi\left(uB - \frac{\varphi}{y}\right) dx \tag{4.4}$$

The necessary conditions of maximum N were obtained in /4,5/. It was shown in /4/ for transonic flow in MHD-generator, that the optimal form channel widens at the greatest possible angle from the inlet to the outlet cross section $y(x) = k_2 x + 1$, and the optimal distribution of the magnetic field intensity consists of a section of the limit extremum where $B(x) \equiv 1$ (close to cross section $x = 0$) and adjoining it the section of two-sided extremum, where $B(x)$ falls from unity to some lower value at the channel outlet. The electrodes were assumed continuous throughout the channel, i.e. $\varphi = \text{const}$. The gain in power due to optimization only of the magnetic field intensity was 8.3% for $\Delta = 2$. The comparison was made with a channel of optimal form with magnetic field intensity $B(x) \equiv 1$ over the whole channel length.

Here we consider a channel with ideally subdivided electrodes, and solve the problem using the system of Eqs.(1.1) and (4.1) with boundary conditions (4.2) and the conjugate system (3.1) with boundary conditions obtained by the known methods of variational calculus. For the considered here problem we have

$$\mu_1 = 0, \quad \mu_2 u + \mu_3 \gamma p = 0 \quad (x = x_k)$$

It was assumed in calculations that $\Delta = 2, \sigma \equiv 1, \gamma = 5/3, u_0 = 0.5, p_0 = 0.0974, \rho_0 = 0.649$ which corresponds to a Mach number $M_0 = 1$ at $x = 0$. The maximum admissible angle of slope of channel wall to the x axis is 20° which yields $k_2 = 3.64$ when $l^\circ/y^\circ(0) = 10$.

The optimal distribution of $\varphi(x)$ is given by the formula

$$\varphi = \frac{By}{2\rho}\left[\rho u - \frac{\mu_2}{y + (\gamma - 1)\mu_3}\right]$$

The optimal distribution of $\varphi(x)$ in the absence of constraints on φ, and $B \equiv 1$ is represented in Fig.1, where the optimal distribution of $B(x)$ is given with $\varphi = \text{const}$ obtained in /4/.

The distribution of power along the channel of the MHD-generator of optimal form $y(x) = 3.64x + 1$ is also shown in Fig.1 for the optimal distribution of $\varphi(x)$ and $B \equiv 1$ (line 1), for the optimal $B(x)$ and $\varphi = \text{const}$ (line 2), and for $B(x) = 1$ and $\varphi = \text{const}$ (line 3). These curves show that the gain in power due to optimization of $\varphi(x)$ is equal 33%, when compared with a generator with compact electrodes when $B \equiv 1$, and 21% when compared with a generator where $B(x)$ is distributed in an optimal manner and the electrodes are compact. In channels with compact electrodes the value of $\varphi = \text{const}$ was in each case optimal. Thus, for example, for line 3 the value $\varphi = 1.2$ was selected.

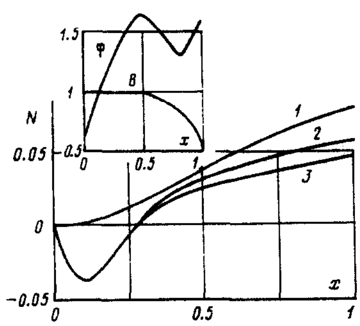

Fig.1

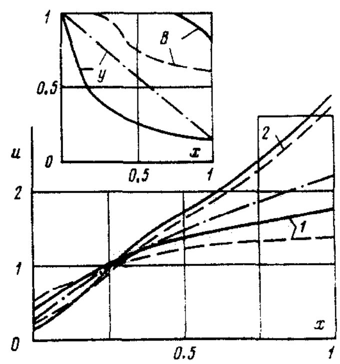

Fig.2

The substantial gain in power, when optimizing $\varphi(x)$ is due to the elimination of the section with energy supply to the gas near the cross section of the channel, which is indicated by the curves of power in Fig.1. For optimal distributions of $\varphi(x)$ and $B(x)$ the point of transition through the speed of sound was shifted along x toward the channel outlet cross section. The optimal channel length was the same as in /4/ and equal its maximum admissible value.

Let us consider the variable problem of constructing an MHD-accelerator. Since in this case solutions in which gas is accelerated from low to possibly high velocities are of interest, we optimize from all modes of flow in the accelerator, the mode with transition through the speed of sound, in the singularity of the saddle type. The flow of conducting medium in a MHD-accelerator is defined by Eqs.(1.1) and (4.1). The boundary conditions at the channel intake are specified by the relations

$$\rho_0 = (1 - u_0^2)^{1/(\gamma-1)}, \quad p_0 = \frac{\gamma - 1}{2\gamma} (1 - u_0^2)^{\gamma/(\gamma-1)}$$

which assume the absence of losses in the flow at $x \leqslant 0$. In the channel outlet cross section a supersonic flow obtains, hence no conditions at $x = x_b$ are specified.

We select as the integral characteristic, that is to be optimized, the velocity u_b at the channel outlet cross section. The control parameters in this problem are $y(x), B(x), \varphi(x)$ and x_b satisfy the inequality (4.3). The qualitative investigation of the steady Eqs.(1.1) and (4.1) had shown that in the mode energy supply to the gas a singularity of the saddle type occurs.

In this case the boundary conditions (one at $x = 0$ and three at $x = x_b$) for integration of Eqs.(3.1) for the Lagrange multipliers are of the form

$$\rho\mu_1 + \gamma p\mu_3 = 0 \quad (x = 0)$$
$$\mu_1 = 0, \quad 1 + u\mu_2 + \gamma p\mu_3 = 0, \quad \mu_2 + \rho u\mu_3 = 0 \quad (x = x_b)$$

We assume that the electrodes are compact, i.e. $\varphi = \text{const}$ throughout the channel. Formulas for the optimal distribution of magnetic field intensity $B(x)$ and the areas of the cross section of the plane channel $y(x)$ on sections of the bilateral extremum at $\sigma \equiv 1$ in this problem are of the form

$$B = \frac{\varphi}{2uy}\left[1 - \frac{(\gamma - 1)\rho u\mu_3}{\mu_2 - (\gamma - 1)\rho u\mu_3}\right] \tag{4.5}$$

$$ay^2 + by + c = 0 \tag{4.6}$$

$$a = \gamma\rho uB^2\{\mu_2(p - \rho u^2) + \mu_3\gamma u[(\gamma - 1)p + \rho u^2]\}$$
$$b = \varphi B\{\mu_2[\gamma p - 2(\gamma - 1)\rho u^2] + 2\mu_3(\gamma - 1)\rho u[\gamma p + (\gamma - 1)\rho u^2]\}$$
$$c = (\gamma - 1)\rho\varphi^2\{\mu_3[\gamma p - (\gamma - 2)\rho u^2] - \mu_2 u\}$$

Flows with $\varphi_2 = 1.5, \gamma = {}^5/_3, \sigma = 1, k_1 = 3, y(0) = 1$ were calculated as examples:

1) in the channel of the accelerator of optimal form at constant magnetic field intensity $B(x) \equiv 1$;

2) in a channel of constant area of transverse cross section $y \equiv 1$ and optimal distribution of magnetic field intensity,

3) in a channel of optimal form and optimal distribution of the magnetic field.

The results of calculations for $\Delta = 1$ are shown in Fig.2. In all considered cases the optimal values were $x_b = 1, \varphi = \varphi_2$.

Calculations had shown that a plane channel of optimal form must be narrowing, and the maximum value $y' = -3$ is reached on the initial section to which is adjoined the section of bilateral extremum (4.6), where the narrowing of the channel is more smooth. From the two roots of Eq.(4.1) the positive one was selected.

In Fig.2 the solid line represents the optimal channel form $y(x)$ when $B \equiv 1$.

The velocity of gas in the outlet cross section of a channel of optimal form is 2.1 times larger than in a channel of constant cross section, and 1.5 times larger than in a channel that narrows in conformity with the linear law (the dash-dot line in Fig.2). The area of cross section of a straight channel was taken equal to the area of intake cross section of the optimal channel, and the area of intake and outlet cross sections of a narrowing channel were specified equal to respective areas of the optimal channel. The presented data were obtained by comparing the velocities u at $x = 1$. The distribution of $u(x)$ are shown in Fig.2, where the dash lines relate to the medium velocity in a channel of constant cross section $y \equiv 1$ (curve *1*) and in a channel of optimal form (curve *2*), and the dash-dot line shows the velocity in a linearly narrowing channel.

As parameter Δ is increased, the gain of velocity u_b increases due to the optimization of the channel form.

The results of calculation of optimal distribution of the magnetic field are also shown in Fig.2.

The optimal distribution of the megnetic field intensity, as in the optimization of MHD-generator consists of two sections, one with maximum admissible intensity $B = 1$ at the beginning of the channel, and is followed by a section of the bilateral extremum (4.5).

In Fig.2 the dash line denotes the optimal distribution $B(x)$ in a channel of constant cross section $y = 1$ and the solid line shows it in the channel of optimal form with simultaneous optimization of $B(x)$ and $y(x)$. The comparison of these two curves shows that the length of the bilaterial extremum section in the channel of constant cross section is greater than in a channel of optimal form. The gain in velocity due to optimization of $B(x)$ is 16.3% when $\Delta = 1, y(x) \equiv 1$.

The distribution of $u(x)$ in the channel of a MHD-accelerator when $B(x)$ is optimized are shown in Fig.2 by solid lines, viz.*1* in a channel of constant cross section and *2* in a channel of optimal form. The optimal distribution of $y(x)$ with simultaneous optimization of $y(x)$ and $B(x)$ virtually does not differ when $\Delta = 1$ from the curve for $y(x)$ shown in Fig.2. With simultaneous optimization of $y(x)$ and $B(x)$ the gain in velocity is 116% when compared with the accelerator in which $B \equiv 1, y \equiv 1$.

These examples show that the optimal form of the MHD-accelerator when Δ is close to unity, yields a substantially greater gain of the medium velocity in the outlet cross section, than the optimization of the distribution of the magnetic field intensity. In an optimal channel the point of transition through the speed of sound shifts toward the outlet cross section.

As parameter Δ is increased, the gain in velocity due to optimization of $B(x)$ increases which is confirmed by data obtained for strong electromagnetic fields /7/.

As the third example we consider the problem of optimization of a constant current plasmatron with a lengthwise blown arc. The statement and solution of the problem of optimization of such plasmatron for subsonic flows are given in /8/, while in /9/ is presented a qualitative investigation of respective steady equations. It is shown in /9/ that a stable transition through the speed of sound form subsonic to supersonic velocity occurs in widening part of a channel and in a singular point of the saddle type.

In many gasdynamic experiments the establishment is required of a plasmatron of such construction in which the gas velocity at the outlet cross section of the channel exceeds the speed of sound. Here, we consider the problem of constructing an axisymmetric plasmatron channel of optimal form for a given Mach number at the outlet.

The flow in such channel is defined by the system of Eqs.(1.1) under the conditions

$$f_1 = f_2 = 0, \quad f_3 = \frac{\eta I^2}{\sigma r^2}, \quad y = r^2, \quad \eta = \frac{I^{\circ 2} l^\circ \sqrt{\rho^\circ}}{\sigma^\circ r^{\circ 2} p^{\circ 3/2}} \tag{4.7}$$

where $I = \text{const}$ is the arc current strength r is the radius of the axisymmetric channel, η is the dimensionless number, and the small circle superscript denotes dimensional quantities.

For integrating the system of Eqs.(1.1) and (4.7) we specify the gas flow rate $\rho u y = m_0$ and its enthalpy $h = h_0$ at $x = 0$.

The controlling parameters of the system are the shape of the channel $r(x)$ and its length. As shown in /8/, the optimal value h_0 is equal to the least possible value, which we take as equal unity in dimensionless form.

As the optimization characteristic we select the power of the plasmatron

$$N = \int_0^{x_b} \frac{\eta I^2}{\sigma r^2} dx$$

The boundary conditions for the system of Eqs.(3.1) are in this case

$$\mu_2(\rho u^2 - p) + \mu_3(\gamma - 1)\rho u p = 0 \quad (x = 0)$$
$$\mu_2(\rho u^2 + p) + \mu_3(\gamma + 1)\rho u p = 0, \quad M = M_b$$
$$\mu_1 \rho^2 u + \mu_2 p + \mu_3 \rho u p = 0 \quad (x = x_b)$$

Calculations were carried out for $\gamma = 5/3$, $k_{1,2} = 3$, $\sigma = 0.25 p/\rho$, $h_0 = 1$, $m_0 = 0.1653$ with allowance for supplementary constraint /8/ on the radius of the channel cross section of the form $r(x) \geqslant r_1$, $r_1 = 0.4$.

It appears that the optimal channel form consists of three sections, viz. the initial narrowing conical, whose angle of narrowing is equal to the maximum admissible, a central cylindrical one with the area of cross section is $r = r_1$, and the finite section of widening conically, the angle of widening is equal to the maximum admissible.

Transition through the speed of sound occurs in the channel cross section where the final conical section adjoins the cylindrical one, the singular point that defines the flow in such channel is a deformed saddle.

The comparison of results of calculation of flow in an optimal channel, or channels first of which consists of two conical sections: initial narrowing one and a widening one adjoining it, and the second which has the initial cylindrical section followed by a widening conical section, has shown that optimization of the channel form only yields in this case considerable gain in power.

For instance, with the dimensionless similarity criterion $\eta = 5$ the power in the optimal channel is by 16% higher than in the first, and by 82% higher than in the second for the same channel lengths, current strength in the arcs, the angles of narrowing and widening, which in all cases were selected equal to maximum admissible, and the area of intake cross sections and Mach number at the outlet $(M_b = 1.85)$.

Results of these calculations are shown in Fig.3, where the channel profiles and the corresponding to them curves of power distribution are denoted by numerals *1, 2, 3*.

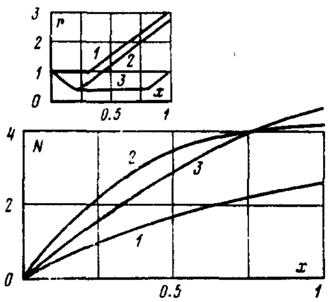

Fig.3

The calculations were carried out by the method of establishment in time for system (1.1) using the scheme proposed in /10/, and for system (3.1) by the three-point difference scheme.

Comparison of solutions obtained using the method of establishment and by integration of ordinary differential equations by the Runge-Kutta method and ranging of the deficient parameters using the Newton method had shown a good agreement of results.

The presented examples have, thus, demonstrated that the proposed here method of solving problems of optimal selection of parameters controlling the transonic flows is applicable for any steady flow with transition through the speed of sound. We would also point out that these methods may be used for optimizing flows with shock waves and can be extended over equation of a more general form that enable the taking into account the phenomenon of relaxation.

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